

No Spontaneous Breakdown of Chiral Symmetry in Nambu-Jona-Lasinio Model

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We argue that the spontaneous breakdown of symmetry in the chirally symmetric Nambu–Jona-Lasinio model which was supposed to illustrate the origin of the low mass of pions in hadron physics does not occur due to strong fluctuations in the σ - π field space. Although quarks acquire a constituent mass, σ and π turn out to have equal heavy masses of the order of the constituent quark mass.

I. INTRODUCTION

The chirally symmetric Nambu–Jona-Lasinio model [1] was the first theoretical laboratory to illustrate how light pions arise from a spontaneous breakdown of chiral symmetry in hadron physics. The first realistic formulation of the model which included flavored quarks, possessed chiral symmetry $SU(3) \times SU(3)$, and a spectrum of σ, π, ρ, A_1 mesons and their $SU(3)$ partners, was formulated and investigated in 1976 by one of the authors [2], and has been the source of inspiration for many papers in nuclear physics in the past twenty years [3]. By eliminating the Fermi fields in favor of a pair of collective scalar and pseudoscalar fields σ and π , as well as vector and axial vector mesons, a Ginzburg Landau like collective field action was derived. This had been studied in detail earlier as an effective action guaranteeing all low-energy properties of hadronic strong interactions which were known from current algebra and partial conservation of the axial current (PCAC).

In two important respects, however, the model was unsatisfactory. First it was not renormalizable in four dimensions, but required a momentum space cutoff Λ to produce finite results. Moreover, to obtain physical quantities of the correct size, the cutoff had to be rather small, below one GeV, thus limiting the reliability of the predictions to very low energies. Second, if the fermions were identified with quarks, the model could not account for their confinement.

The nonrenormalizability was removed in [2] by replacing the four-fermion interaction by the exchange of a massive vector meson. The different attractive meson channels were obtained by a Fierz transformation. The mass of the vector meson took over the role of the cutoff. The energy range of applicability was, however, not increased since the model would still allow for free massive quarks.

The purpose of this note is to point out a much more severe problem with the model which seems to invalidate most conclusions derived from it in the literature: If chiral fluctuations are taken into account in a certain nonperturbative approximation, the spontaneous symmetry breakdown disappears, and the zero-mass pions acquire the same mass as the σ -mesons, both of the same order as the constituent quark mass. The nonperturbative nature of the argument seems to be the reason why the phenomena has been overlooked until now.

Since the Nambu–Jona-Lasinio model is incapable of accounting for confinement, it gave no reason for introducing colored quarks. It is curious to observe that the restoration of symmetry by chiral fluctuations would offer such a reason, albeit with an unphysical number of colors: the physically desired spontaneous symmetry breakdown conclusion can only be achieved by introducing at least five identical replica of fermions. The existing three colors are insufficient to save the purpose of the model.

The non-perturbative arguments used in this paper are analogous to those applied before in a discussion of the Gross-Neveu model [4] in $2 + \varepsilon$ dimensions [5], where it was shown that this model has two phase transitions, one where quarks become massive and another one where chiral symmetry breaks spontaneously. They have also been applied to explain the experimental observation of two transitions in high- T_c superconductors [6], and to show that directional fluctuations in Ginzburg-Landau theories with spontaneously broken $O(N)$ symmetry disorder the system *before* size fluctuations of the order field become relevant [7].

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II. NAMBU–JONA-LASINIO MODEL

Let us briefly recall the relevant features of the Nambu–Jona-Lasinio model for our considerations. The model contains N_f quark fields $\psi(x)$, one for each flavor. Each of them may appear with N_c colors, such that the total number of quarks is $N = N_f \times N_c$. Since the fluctuation phenomenon to be discussed will be caused by the almost massless modes, we may restrict ourselves to the almost massless up and down quarks. We will comment later on the effect of the heavier quarks.

The Lagrangian of the model is given by [3]

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m_0)\psi + \frac{g_0}{2N_c} \left[(\bar{\psi}\psi)^2 + (\bar{\psi}\lambda_a i\gamma_5 \psi)^2 \right], \quad (1)$$

where an implicit summation over $a = 1, 2, 3$ is assumed. A small diagonal quark mass matrix m_0 breaks slightly the $SU(2) \times SU(2)$ part of the chiral symmetry which lifts the mass of the pion to a small nonzero value. We have omitted the flavor symmetric vector gluon exchange used in Ref. [2] which would have given rise, after a Fierz transformation, to additional vector and axial vector interactions, which would not influence the chiral fluctuations to be investigated here. Thus we use the original nonrenormalizable interaction corresponding to an infinite vector gluon mass. The coupling constant in (1) is defined with the number of colors N_c in the denominator, to allow for a finite $N_c \rightarrow \infty$ limit of the model at a fixed g_0 . The 2×2 -dimensional matrices $\lambda_a/2$, with $a = 1, \dots, 3$ generate the fundamental representation of flavor $SU(2)$, and are normalized by $\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$.

Via a Hubbard-Stratonovich transformation, the Lagrangian (1) is converted into a theory of collective scalar and pseudoscalar fields σ and π_a . Defining the propagator in the presence of the meson fields

$$G \equiv \frac{i}{i\cancel{\partial} - m_0 - \sigma - i\gamma_5 \lambda_a \pi_a}, \quad (2)$$

and adding external quark sources $\eta, \bar{\eta}$, one can integrate out the quark fields from the corresponding Lagrangian. Summing over colors, the generating functional of the Green functions takes the well-known form

$$\mathcal{Z} = \int \mathcal{D}\sigma \mathcal{D}\pi \exp \left\{ iN_c \left[-i\text{Tr}' \ln iG^{-1} - \frac{1}{2g_0} \int d^D x (\sigma^2 + \pi_a^2) + iN_c^{-1} \int d^D x d^D y \bar{\eta} G \eta \right] \right\}. \quad (3)$$

The term inside the brackets is the collective field action $\mathcal{A}[\sigma, \pi]$, whereas the symbol Tr' includes both, the functional spacetime “index” x , and the internal trace over spin and flavor indices: $\text{Tr}' \equiv \int d^D x \text{tr}_\gamma \text{tr}_f$.

By extremizing $\mathcal{A}[\sigma, \pi]$ at zero sources $\eta, \bar{\eta}$, we obtain the field equation for the collective field (σ, π_a) :

$$\text{tr}_\gamma \text{tr}_f \left[G(x, x) \begin{pmatrix} 1 \\ i\lambda_a \gamma_5 \end{pmatrix} \right] = \frac{1}{g_0} \begin{pmatrix} \sigma(x) \\ \pi_a(x) \end{pmatrix}. \quad (4)$$

For constant fields, this equation becomes a *gap equation*. Its solutions will be marked by a superscript “s” for “stationary phase approximation”. From now on, unless explicitly stated, we shall consider the model with zero mass, $m_0 = 0$. The stationary pseudoscalar solutions π_a^s can always be chosen to be vanishing, while the scalar solutions can be $\sigma^s = 0$, or $\sigma^s \equiv \rho_0$. In the first case, the ground state is chirally symmetric, in the second the symmetry is spontaneously broken. This is the state of physical interest whose stability will now be discussed.

III. EFFECTIVE POTENTIAL AND GAP EQUATION

In the limit $N_c \rightarrow \infty$, the generating functional is given *exactly* by the extremal field configurations, which will be parameterized as $(\sigma^s(x), \pi_a^s(x)) = (\rho_0(x), \mathbf{0})$. The system has an effective action per quark

$$\frac{\Gamma(\rho_0, \Psi, \bar{\Psi})}{N_c} = -i\text{Tr}' \ln iG_{\rho_0}^{-1} - \frac{1}{2g_0} \int d^D x \rho_0^2 + \frac{1}{N_c} \int d^D x \bar{\Psi}_a iG_{\rho_0}^{-1} \Psi_a, \quad (5)$$

where $\Psi = iG_{\rho_0}\eta$ is the expectation value $\langle \psi \rangle$ of the quark field, and G_{ρ_0} its propagator

$$G_{\rho_0} = \frac{i}{i\cancel{\partial} - \rho_0}. \quad (6)$$

This shows that the solution of the gap equation with $\rho_0 \neq 0$ describes quarks with a nonzero mass $M = \rho_0$, which has been generated by the spontaneous symmetry breakdown, and is referred to as the *constituent quark mass*. In the present approximation of zero bare mass m_0 , the constituent quark mass is about equal to 300 MeV for up and down quarks (see the discussion in Refs. [2,8]). In either case, the Green function (2) in the stationary field is diagonal in flavor space.

In the absence of external quark sources, the ground state expectation value of a fermion field is always zero, and the expectation value $\rho_0(x)$ is constant, so that (5) reduces into

$$\frac{\Gamma(\rho)}{N_c} = -i\text{Tr}' \ln iG_\rho^{-1} - \frac{1}{2g_0} \int d^D x \rho^2, \quad (7)$$

where we have allowed the fields σ and π_a to be nonextremal, defining $\sigma^2 + \pi_a^2 \equiv \rho^2$, and reserving the notation ρ_0^2 for the extremum. This is determined, after a Wick rotation to euclidean momenta p_E with $p_0 = ip_{E,0}$, $id^D p \rightarrow -d^D p_E$, $p^2 \rightarrow -p_E^2$, by the gap equation

$$\frac{1}{g_0} = 2 \times 2^{D/2} \int \frac{d^D p_E}{(2\pi)^D} \frac{1}{p_E^2 + \rho_0^2}. \quad (8)$$

We have divided the two sides of the gap equation by a common factor ρ_0 , since we want to study the spontaneously broken phase.

The gap equation must be regularized, which may be done in many ways. Here, we shall use two methods: analytic continuation in the dimension D , and a cutoff Λ in momentum space. The former is mathematically more elegant and has the advantage of relating the properties in four dimensions to those in $2 + \varepsilon$. It has, however, some unphysical properties which require special attention, as we shall see. Such problems are absent in a cutoff regularization scheme, which exhibits clearly the physical divergences caused by the infinite number of degrees of freedom of the field system. Factorizing the integral in (8) into direction and size of the momentum p_E , we bring the gap equation to the form

$$\frac{1}{g_0} = 2\rho_0^{D-2} \frac{\Gamma(1 - D/2)}{(2\pi)^{D/2}}. \quad (9)$$

Denoting by Ω the D -dimensional volume $\int d^D x$, the volume density $v(\rho) \equiv -\Gamma(\rho)/\Omega$ of the effective action (7) is the *effective potential* per quark. Performing the internal traces, and subtracting a divergent constant term associated with the chirally symmetric state with $\rho_0 = 0$, we obtain the *condensation energy* in euclidean space:

$$\Delta v(\rho) = \frac{N_c}{2} \left[\frac{1}{g_0} \rho^2 - \rho^D \frac{4}{D} \frac{1}{(2\pi)^{D/2}} \Gamma(1 - D/2) \right]. \quad (10)$$

In an even number of dimensions D , both the gap equation (9) and the effective potential (10) are divergent, due to a pole in the factor $\Gamma(1 - D/2)$. Introducing the diverging parameter $b_\epsilon = 2\Gamma(1 - D/2)/[D(2\pi)^{D/2}]$, we can rewrite the gap equation and effective potential in the more compact form as

$$\frac{1}{g_0} = D\rho_0^{D-2} b_\epsilon, \quad (11)$$

$$\Delta v(\rho) = \frac{N_c}{2} \left[\frac{1}{g_0} \rho^2 - 2\rho^D b_\epsilon \right]. \quad (12)$$

In the more physical regularization with a cutoff Λ in momentum space, these expressions look more complicated:

$$\frac{1}{g_0} = \frac{2}{(2\pi)^2} \left[\Lambda^2 - \rho_0^2 \ln \left(1 + \frac{\Lambda^2}{\rho_0^2} \right) \right], \quad (13)$$

$$\Delta v(\rho) = \frac{N_c}{2} \left\{ \frac{1}{g_0} \rho^2 - \frac{2}{(2\pi)^2} \left[\frac{\rho^2 \Lambda^2}{2} + \frac{\Lambda^4}{2} \ln \left(1 + \frac{\rho^2}{\Lambda^2} \right) - \frac{\rho^4}{2} \ln \left(1 + \frac{\Lambda^2}{\rho^2} \right) \right] \right\}. \quad (14)$$

The results (11) and (12) of the analytic regularization scheme can be mapped roughly into the cutoff results (13) and (14) if we recall the special property of dimensional regularization that all integrals over pure momentum powers vanish identically: $\int d^D k (k)^\alpha = 0$ (Veltman's rule). Thus, arbitrary pure powers of the cutoff $\Lambda^{\alpha+D}$ have no counterpart in dimensional regularization. Only logarithmic divergences can be related to diverging pole terms $1/\epsilon \rightarrow \infty$ for $\epsilon \rightarrow 0$. It is therefore inconsistent to relate ϵ to Λ by setting $\Gamma(\epsilon/2 - 1) \approx \Lambda^2/\rho_0^2$, as proposed by Krewald and Nakayama [9]. Only the logarithmic divergence in (13) can be mapped to the small- ϵ divergence in

(11), setting $\Gamma(\epsilon/2 - 1) \approx -\ln(1 + \Lambda^2/\rho_0^2)$. With their inconsistent identification, Krewald and Nakayama matched Λ by an $\epsilon > 2$ which lies in the wrong region $D < 2$, the physically relevant range being $D \in (2 + \epsilon, 4 - \epsilon)$. Note that the matching of the logarithm at the level of the effective potential leads to the properly matched gap equation, thus having circumvented the unphysical properties of the analytic regularization. The free use of this scheme in renormalizable field theories relies on the fact that all infinities are eventually absorbed in unobservable bare quantities, such that the artificial zeros of the integrals over pure powers of momenta cannot produce problems. In nonrenormalizable theories, on the other hand, only a cutoff (or a related Pauli-Villars regularization) is physical, and analytic regularization must be treated with caution. This is seen even more dramatically in integrals which do not have logarithmic infinities. For example the condensation energy (10) in $D = 3$ dimensions would be a finite negative number in analytic regularization, while being a linearly divergent positive function of the cutoff.

IV. CHIRAL FLUCTUATIONS

Since the physical number of quarks N_c is finite, the fields perform fluctuations of magnitude $1/\sqrt{N_c}$ around their extremal value. As long as N_c can be considered as a large number, the deviation from the extremal field configuration $(\sigma', \pi'_a) \equiv (\sigma - \rho_0, \pi_a)$ are small, and the action can be expanded in powers of (σ', π'_a) . The quadratic terms in this expansion define the propagators of the collective fields (σ', π'_a) . The higher expansion terms of the trace of the logarithm in (3) define the interactions. With this decomposition, the inverse of the quark propagator (2) can be decomposed into a constant and a fluctuating part, setting $iG^{-1} = iG_{\rho_0}^{-1} - (\sigma' + i\gamma_5 \lambda_a \pi'_a)$, with G_{ρ_0} of Eq. (6). Then we have

$$\text{Tr} \ln iG^{-1} = \text{Tr} \ln iG_{\rho_0}^{-1} + \text{Tr} \ln [1 + iG_{\rho_0} (\sigma' + i\gamma_5 \lambda_a \pi'_a)]. \quad (15)$$

An expansion of the last term up to the second order in the fields gives an approximate partition function (with $\mathcal{Z}_0 \equiv \exp[-\Omega_E N_c v(\rho_0)]$ and Ω_E is the euclidean volume)

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}\sigma \mathcal{D}\pi \exp \left(iN_c \left\{ \frac{i}{2} \text{Tr}' [iG_{\rho_0} (\sigma' + i\gamma_5 \lambda_a \pi'_a)]^2 - \frac{1}{2g_0} \int d^D x (\sigma'^2 + \pi_a'^2) \right\} \right). \quad (16)$$

The functional matrix between the fields in the exponent gives us directly the inverse of the desired collective free field propagators $G_{\sigma'}, G_{\pi'}$. In momentum space, we identify

$$\mathcal{A}_0[\sigma', \pi'] = \frac{1}{2} \int d^D q [\pi'_a(q) G_{\pi'}^{-1} \pi'_a(-q) + \sigma'(q) G_{\sigma'}^{-1} \sigma'(-q)], \quad (17)$$

where

$$G_{\sigma, \pi}^{-1} = 2 \times 2^{D/2} N_c \int_0^1 dy \int \frac{d^D p_E}{(2\pi)^D} \frac{q_E^2 + p_E q_E + (2\rho_0^2, 0)}{[(q_E^2 + 2p_E q_E)y + p_E^2 + \rho_0^2]^2}. \quad (18)$$

In this expression, the gap equation (8) has been used to eliminate the term $1/g_0$. The notation $(2\rho_0^2, 0)$ indicates that only the equation for σ contains an extra term $2\rho_0^2$.

In four spacetime dimensions, the integral evaluated in dimensional regularization reduces to $q_E^2/2$ for the pseudoscalars, and to $(q_E^2 + 4\rho_0^2)/2$ for the scalars, both with a diverging coefficient. The first leads to a zero mass for pions as a manifestation of Goldstone's theorem, the second to a mass equal to twice the constituent quark mass for the σ -mesons. For a finite result, the integrals must be regularized. In $D = 4 - \epsilon$ dimensions, the inverse euclidean propagator is seen to start out for small q_E^2 like

$$G_{\pi}^{-1} \approx N_c \left(1 - \frac{D}{2} \right) D b_{\epsilon} \frac{1}{\rho_0^{4-D}} \frac{q_E^2}{2} \equiv Z_{\pi}^{(\epsilon)}(\rho_0) q_E^2 + \mathcal{O}(q_E^4). \quad (19)$$

with the same b_{ϵ} as defined above Eq. (11). If the theory is regularized with a cutoff Λ in momentum space, this becomes

$$G_{\pi}^{-1} = \frac{N_c}{(2\pi)^2} \left[\ln \left(1 + \frac{\Lambda^2}{\rho_0^2} \right) - \frac{\Lambda^2}{\Lambda^2 + \rho_0^2} \right] q_E^2 \equiv Z_{\pi}^{(\Lambda)}(\rho_0) q_E^2. \quad (20)$$

In the right-hand part of the two equations, the factors in front of q_E^2 have been identified as the wave function renormalization constants $Z_{\pi}(\rho_0)$ of the pion field in the two regularization schemes.

As a consequence of the spontaneous symmetry breakdown, the fluctuations of the pseudoscalar fields are massless. These fields appear in the x -space version of the action (17) in a pure gradient form

$$\mathcal{A}_0[\pi'] = \frac{\beta}{2} \int d^D x \{ [\partial \pi'_a(x)]^2 \}, \quad (21)$$

with $\beta = Z_\pi$. Due to chiral symmetry, this gradient action can be extended to the gradient action of an arbitrary field (σ, π_a) . Introducing the directional unit vector fields $n_i = (\hat{\sigma}', \hat{\pi}'_a) \equiv (\sigma', \pi'_a)/\rho$, we find:

$$\mathcal{A}_0[n_i] = \frac{\beta(\rho^2)}{2} \rho^2 \int d^D x (\partial n_i)^2, \quad i = 1, \dots, N_n, \quad (22)$$

with $N_n = 4$ and

$$\beta(\rho) = Z_\pi(\rho). \quad (23)$$

This chirally invariant action describes the massless pions with *all* multipion interactions.¹ The prefactor β is called the *stiffness of the directional fluctuations* [5,7,10–12]. In analytic regularization, the result (19) shows that the stiffness of pion fluctuations in $D=2$ dimensions becomes

$$\beta = \frac{N_c}{2\pi\rho_0^2}, \quad (24)$$

thus coinciding with the stiffness calculated in Ref. [5] in the Gross-Neveu model (which contained a factor N to be identified with the present $N_f \times N_c = 2N_c$).

With the more physical cutoff regularization in $D = 4$ dimensions, the stiffness of directional fluctuations is

$$\beta = Z_\pi^{(\Lambda)}(\rho_0) = \frac{N_c}{(2\pi)^2} \left\{ \ln \left[1 + \left(\frac{\Lambda}{\rho_0} \right)^2 \right] - \frac{\Lambda^2}{\rho_0^2 + \Lambda^2} \right\}. \quad (25)$$

This is the crucial quantity leading to our fatal conclusions for the restoration of chiral symmetry. The stiffness (25) is far too small to let the directional field settle in a certain direction, required for spontaneous symmetry breakdown. The disordering effect of phase fluctuations is well-known from many model studies of the $O(4)$ -symmetric Heisenberg model on a lattice. High-temperature expansions and Monte Carlo simulations have shown that there exists a critical stiffness below which the system goes over into a disordered state.

For an analytic estimate of the critical stiffness, we relax the unit vector constraint for the vectors n_i in (22) by introducing an additional field $\lambda(x)$ playing the role of a Lagrange multiplier. The n_i -fields can then be integrated out in the partition function, leading to an action

$$S = \frac{N_n}{2} \text{Tr} \ln [-\partial^2 + \lambda(x)] - \beta(\rho^2) \rho^2 \int d^D x \frac{\lambda(x)}{2}, \quad (26)$$

where Tr denotes the functional trace (the summation over the fields component has already been performed). For a large number N_n of components, the fluctuations are suppressed, and the field $\lambda(x)$ becomes a constant satisfying a second gap equation

$$\beta = \frac{N_n}{\rho^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + \lambda}. \quad (27)$$

If there is a nonzero solution $\lambda \neq 0$, this will play the role of a square mass of the n_i -fluctuations, and represents an order parameter in the directional phase transition. The model has a phase transition at a critical stiffness

$$\beta_c = \frac{N_n}{\rho^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2}. \quad (28)$$

¹Only two approximations are involved: the first one consists in freezing the size ρ of the fluctuations. The second one neglects corrections due to the finiteness of the sigma mass. The latter corrections are expected to be of the order $f_\pi^2/(4M^2) \approx 3\%$.

For a smaller stiffness, the phase fluctuations are so violent that the system goes into a disordered phase with $\lambda \neq 0$ giving all fields n_i a nonzero square mass λ . Since the fields n_i are the normalized σ and π_a fields of the model, this determines an equal nonzero square mass of σ and π_a mesons, and thus a restoration of chiral symmetry.

Note that the quarks are still massive: their constituent mass is a consequence of the *formation* of the pairs, which are strongly bound for small N_c . The phase transition taking place at the critical value of the stiffness, on the other hand, is related to the Bose-Einstein condensation of the pairs. At small N_c , the two processes are widely separated. This separation of the two transitions (pair formation and pair condensation) can be judged by the simple fluctuation criterion in Ref. [7].

In our model, the number N_n is equal to four, which is not very large. Fortunately, Monte Carlo studies of the model [13–15] have shown that $N_n = 4$ is large enough to ensure the existence of the transition and the quantitative reliability of the theoretical estimate of the critical stiffness (28). From an evaluation of (28) *on a lattice*, and a comparison with Monte Carlo studies, we estimate that the critical stiffness obtained from (28) is correct to within less than 2% [16,14] or 6% [16,13,15]. The same maximal error is expected if we work in the continuum using a momentum cutoff scheme.

For $N_n = 4$ and a cutoff Λ_π in the integral (28) over pion momenta, the critical stiffness is given by

$$\beta_c = \frac{4}{16\pi^2} \frac{\Lambda_\pi^2}{\rho^2}. \quad (29)$$

By comparing this with the stiffness of the model in (25), we find

$$N_c = \left(\frac{\Lambda_\pi}{\Lambda}\right)^2 \left(\frac{\Lambda}{\rho_0}\right)^2 \left\{ \ln \left[1 + \left(\frac{\Lambda}{\rho_0}\right)^2 \right] - \frac{(\Lambda/\rho_0)^2}{1 + (\Lambda/\rho_0)^2} \right\}^{-1}. \quad (30)$$

This equation determines the number N_c of identical quarks which is necessary to produce a large enough stiffness β to prevent the restoration of chiral symmetry. Only if the number of colors exceeds this critical value, will the model possess a phase in which the pion is a massless Goldstone boson, and σ a meson with a mass twice as large as that of the constituent quarks. The critical number (30) is plotted as the solid curve in Fig. 1 for $\Lambda_\pi = \Lambda$. We see that $N_c = 5$ would be the smallest allowed value. This number, however, is incompatible with color SU(3). This suggests that the Nambu–Jona-Lasinio model always remains in the symmetric phase, due to chiral fluctuations. It can therefore not be used to describe the chiral symmetry breakdown of quark physics, as has been claimed by many publications, which have appeared in particular in nuclear physics [3].

Can this conclusion be avoided by a different choice of parameters? To obtain a critical value smaller than $N_c = 3$ would require a pionic cutoff $\Lambda_\pi \lesssim 0.8\Lambda$. However, the cutoff cannot be chosen at will. Let us study the cutoff dependence more precisely. For this, we refine the previous crude estimate (28), (29) of the critical stiffness, which will henceforth be called Approximation 1, by taking better account of the shorter wavelength fluctuations, replacing the action (22) by

$$\mathcal{A}_1[n_i] = \frac{\rho^2}{2} \int d^D x n_i(x) G_\pi^{-1}(-\partial^2) n_i(x), \quad (31)$$

with $G_\pi^{-1}(-\partial^2)$ from Eq. (18). This exchanges $1/k^2$ in Eq. (28) by the full pion propagator $G_\pi(k^2)/Z_\pi^{(\Lambda)}$ associated with the action (31). The cutoff Λ_π makes the integral over pion momenta finite. Its size is fixed by physical considerations. The pion fields in the symmetry-broken phase are composite, and will certainly not be defined over length scales much shorter than the inverse binding energy of the pair wave function, which is equal to $2M = 2\rho_0$. Thus we perform the integral in the modified equation (28) up to the cutoff $4M^2$. This is Approximation 2, yielding the solid curve in Fig. 3.

The phase with broken symmetry for three colors would be reached only if the quark loop integration is cut off at $\Lambda^2 \gtrsim 11M^2$. Such a large value, however, is incompatible with the experimental value of the pion decay constant $f_\pi \approx 0.093$ which is given, in the large- N_c limit of the model, by $f_\pi/M = Z^{1/2}(M)$. For typical estimates of constituent quark masses $M \in (300, 400)$ MeV [2], we find that Λ^2/M^2 should lie in the range (3.3, 7.3), the highest value corresponding to the lowest possible mass 300 MeV.

The above study has given us only the critical point, where the pion mass goes to zero. We can do more and determine the common nonzero square masses $m_\sigma^2 = m_\pi^2 = \lambda$ of σ and π_a -fields in the phase of restored chiral symmetry. This is the subject of the next section.

V. MESON MASSES

The chiral fluctuations give rise to a change of the effective potential. They add to $\Delta v(\rho)$ in Eq. (14) an additional energy coming from the stationary point of the action (26) at a constant $\lambda(x) = \lambda$:

$$\Delta'_1 v(\rho, \lambda) = -\frac{1}{2}\lambda Z_0 \rho^2 + \frac{N_n}{2} \int_0^{\Lambda_\pi^2} \frac{dq_E^2 q_E^2}{16\pi^2} \ln[q_E^2 + \lambda], \quad (32)$$

$$\Delta'_2 v(\rho, \lambda) = -\frac{1}{2}\lambda Z(\rho) \rho^2 + \frac{N_n}{2} \int_0^{\Lambda_\pi^2} \frac{dq_E^2 q_E^2}{16\pi^2} \ln[G^{-1}(q_E^2)/Z(\rho) + \lambda], \quad (33)$$

for Apprs. 1 and 2, respectively, where the latter has $-\partial^2$ replaced by $G_\pi^{-1}(-\partial^2)/Z(\rho)$. Extremizing $\Delta v(\rho) + \Delta'_{(1,2)} v(\rho, \lambda)$ yields two coupled gap equations replacing the independent gap equations (13) and (27). Introducing the reduced quantities $\bar{Z}(x) = \ln(1 + x^{-1}) - (1 + x)^{-1}$, and $x \equiv \rho^2/\Lambda^2$, $y \equiv \lambda/\Lambda^2$, we have for Appr. 1:

$$x_0 \ln(1 + x_0^{-1}) + \frac{y}{2} \frac{d}{dx} [x \bar{Z}(x)] = x \ln(1 + x^{-1}), \quad (34)$$

$$N_c x \bar{Z}(x) = \left(\frac{N_n}{4}\right) \left\{ \left(\frac{\Lambda_\pi}{\Lambda}\right)^2 - y \ln \left[1 + \left(\frac{\Lambda_\pi}{\Lambda}\right)^2 y^{-1} \right] \right\}. \quad (35)$$

For Appr. 2, the coupled gap equations are more complicated since the full q^2 -dependence of Z_π has to be taken into account. They read

$$\begin{aligned} x_0 \ln(1 + x_0^{-1}) + \frac{N_n}{8N_c} \left\{ \int_0^{(\Lambda_\pi/\Lambda)^2} k^2 dk^2 \left[\frac{\bar{Z}(x_0)}{\bar{Z}(k^2, x_0)} \right] \frac{d}{dx_0} \left[\frac{\bar{Z}(k^2, x_0)}{\bar{Z}(x_0)} \right] \right\} + \frac{y}{2} \frac{d}{dx} [x \bar{Z}(x)] \\ = x \ln(1 + x^{-1}) - \frac{N_n}{8N_c} \left\{ \int_0^{(\Lambda_\pi/\Lambda)^2} \frac{k^4 dk^2}{k^2 [\bar{Z}(k^2, x)/\bar{Z}(x)] + y} \frac{d}{dx} \left[\frac{\bar{Z}(k^2, x)}{\bar{Z}(x)} \right] \right\}, \end{aligned} \quad (36)$$

$$N_c x \bar{Z}(x) = \left(\frac{N_n}{4}\right) \int_0^{(\Lambda_\pi/\Lambda)^2} \frac{k^2 dk^2}{k^2 [\bar{Z}(k^2, x)/\bar{Z}(x)] + y}, \quad (37)$$

where $\bar{Z}(k^2, x)$ is taken from the pion propagator (18), and is given by

$$\bar{Z}(k^2, x) = \left(\frac{N_c}{2\pi^2}\right) \int_0^1 p^2 dp^2 \int_0^1 \frac{(1-z)dz}{[p^2 + k^2 z(1-z) + x]^2}. \quad (38)$$

There is no need to write down the lengthy analytic solution to this double integral. The coupling constant g_0 has been eliminated in favor of the reduced mass x_0 which characterizes the model uniquely above N_c^{cr} , where $y = 0$ and thus $\lambda = 0$. In that case, Eq. (34) of Appr. 1 reduces to (13). Equations (35) and (37), on the other hand, determine the common square mass λ of σ and π_a as a function of N_c , which begins developing for $N_c < N_c^{\text{cr}}$. Note that going from Appr. 2 to 1 corresponds to using a momentum-independent pion normalization $\bar{Z}(k^2, x) = \bar{Z}(x)$. This makes (36) and (37) coincide with (34) and (35).

The solutions of (34)–(37) are plotted in Figs. 1–4. Figs. 1–2 are for Appr. 1, restricted to the case $\Lambda_\pi = \Lambda$. Qualitatively, the pictures remain the same for different ratios Λ_π/Λ . Quantitatively, there is only a shift in the critical number of color (solid curve of Fig. 1) to $N_c^{\text{cr}} = 3$ as Λ_π/Λ is lowered to 0.8, while it increases above the given curve if $\Lambda_\pi/\Lambda > 1$. This is due to the fact that at the critical point corresponding to $\lambda = 0$, one sees from Eq. (30) (or from Eq. (35) with $y = 0$) that $N_c^{\text{cr}} \propto (\Lambda_\pi/\Lambda)^2$. The dashed curves of Figs. 1 and 2 are explained in the corresponding legends. Here we only remark that the shape of the dashed curves in Fig. 1 can be understood from the gap equations (34) and (35) without solving them, because $x \bar{Z}(x)$ is maximal at the minimum of N_c^{cr} .

Figures 3 and 4 correspond to Appr. 2, in which the full momentum dependence for the pion normalization constant is taken into account, and in which the pionic cutoff is $\Lambda_\pi^2 = 4M^2$, for which we get the ratio $(\Lambda_\pi/\Lambda)^2 = 4x_0$. The solid curve in Fig. 3 gives the critical number of color in this particular case. Although the conclusion is not as strong as in Appr. 1, our result concerning the lack of breaking of chiral symmetry is robust, since the crossing with the line $N_c = 3$ takes place at a cutoff $\Lambda_\pi^2/M^2 \gtrsim 11$, which lies outside of the admissible range (3.3, 7.3) implied by the physical value of the pion decay constant $f_\pi = 93$ MeV, as discussed at the end of the previous section.

Finally, we give in Fig. 5 the stiffness as a function of the number of color for Appr. 1. The three curves depend so weakly on ρ_0 that they seem to coincide. To make the ρ_0 -dependence visible, we have plotted an extra dotted curve for a very small value $\rho_0 = 0.224\Lambda$ (dotted).

Let us emphasize that these conclusions cannot be reached in the dimensional regularization scheme since, as explained at the end of Section 3, the integral in (28) determining the critical stiffness vanishes. Here the unphysical nature of dimensional regularization makes its application impossible.

Before concluding, let us also remark that the cutoff chosen in Appr. 2 is completely different from that in Appr. 1, where the ratio of cutoffs is a constant. In Appr. 2, the ratio of cutoffs is a function of x_0 : $\Lambda_\pi^2/\Lambda^2 = 4x_0$. If we had taken the cutoff in the same way as in Appr. 1 ($\Lambda_\pi^2/\Lambda^2 = 1$), the curve giving the critical number of colors would also have had the same shape as in Appr. 1, although the integration would have been much more involved: the minimum number of color would then be 5.2, whatever the value of Λ^2/ρ_0^2 , a value which is even higher than in Appr. 1. We see that Appr. 2 as presented above, with the physically motivated cutoff $\Lambda_\pi^2 = 4M^2$, gives then the lowest critical number of colors.

Our conclusions were derived from a study of only the σ , π fields. The inclusion of other flavors does not help preventing the restoration since the associated pseudoscalar mesons are quite massive, making their fluctuations irrelevant to the described phenomenon.

VI. CONCLUSION

We have shown that within a certain nonperturbative approximation, the Nambu–Jona-Lasinio model does not really display the spontaneous symmetry breakdown for whose illustration it was constructed. The fluctuations of σ - and π -fields restore chiral symmetry and make σ and π equally massive. If our conclusion survives more refined approximations, this would invalidate a large number of publications, especially in nuclear physics, which have been based on the existence of a symmetry-broken ground state of the model. In particular, all studies of the temperature dependence of the symmetry-broken state [3] would deal with nonexistent objects, thus calling for further investigations. Finally, we note that our no-go result for the Nambu–Jona-Lasinio model does not imply problems with the effective-action approach to chiral dynamics. Certainly, there exists an effective chiral action for the meson sector of quantum chromodynamics which does contain almost massless pions for $N_c = 3$. It is only the Nambu–Jona-Lasinio model as it stands which is incapable of describing these for such a low number of colors. In fact, a recent paper [17] prompted by a first version of our preprint points out that an extension of the Nambu–Jona-Lasinio model by interactions involving higher-dimensional operators is not subject to our no-go theorem. Another escape is possible by adding gradient and quartic interaction terms for σ - and π -fields to the initial action, thus extending the Nambu–Jona-Lasinio model to a linear sigma model [18].

ACKNOWLEDGMENTS

We thank T. Neuhaus for discussions. The work of B. VdB was partially supported by the Institut Interuniversitaire des Sciences Nucléaires de Belgique and by the Alexander von Humboldt Foundation.

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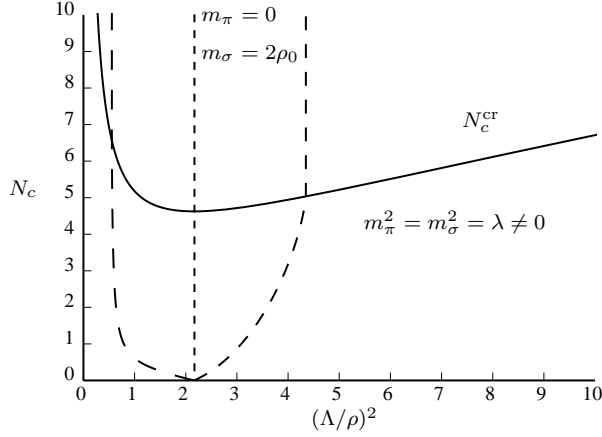


FIG. 1. Solid curve shows *Approximation 1* to critical number of colors N_c^{cr} as a function of the extremal value of $\rho = \sqrt{\sigma^2 + \pi_a^2}$, above which chiral symmetry is restored. The dashed curves indicate the solutions to the two gap equations (34) and (35) for three different values of the constituent quark mass $\rho_0 = M$ in the symmetry-broken phase above N_c^{cr} . The three quark masses lie above ($\rho_0 > \rho^*$), below ($\rho_0 < \rho^*$), and at $\rho^* = M^* \approx \sqrt{0.46}\Lambda$ where N_c^{cr} takes the minimal value 4.62, with a constituent quark mass above N_c^{cr} of 0.678 Λ (short-dashed curve). The medium-dashed curve corresponds to a constituent quark mass 0.479 Λ , and the long-dashed to 1.342 Λ .

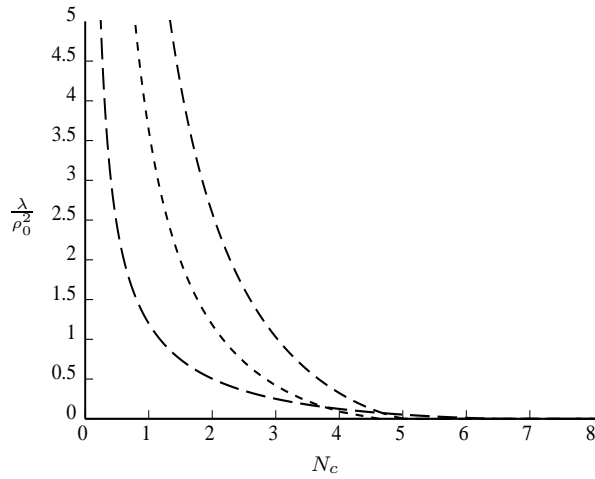


FIG. 2. Common square masses $m_\sigma^2 = m_\pi^2 = \lambda$ as a function of N_c in *Approximation 1*. The three curves start at different critical values N_c^{cr} which can be read off Fig. 1.

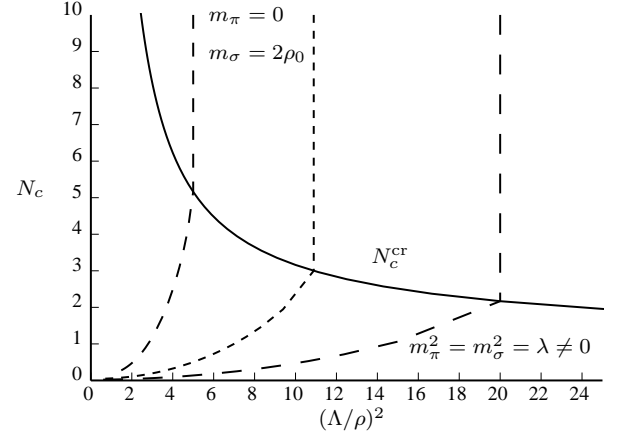


FIG. 3. Same plot as in Fig. (1), but for *Approximation 2*, the dashed curves indicating the solutions of the two gap equations (36) and (37). The three mass values are now $\rho_0 > \rho^*$, $\rho_0 < \rho^*$, $\rho_0 = \rho^*$, with $\rho^* = M^* \approx \sqrt{0.092}\Lambda$, corresponding to the minimal critical number of colors $N_c^{\text{cr}} = 3$, implying a constituent quark mass above $N_c^{\text{cr}} = 3$ of 0.303 Λ (short-dashed curve). The medium-dashed curve are for a constituent quark mass 0.447 Λ , and the long-dashed for 0.224 Λ .

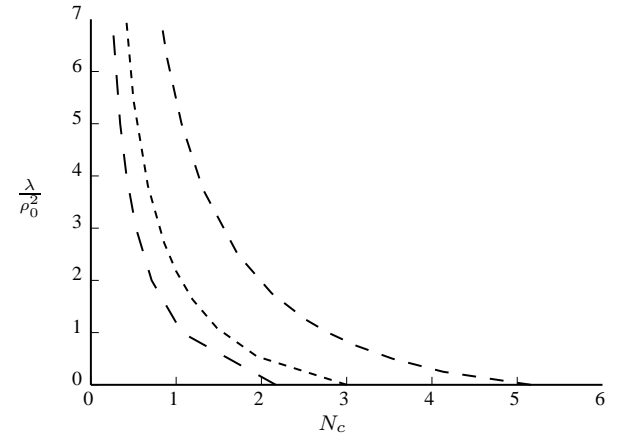


FIG. 4. Common square masses $m_\sigma^2 = m_\pi^2 = \lambda$ as a function of N_c in *Approximation 2*. The three curves start at different critical values N_c^{cr} which can be read off Fig. 3.

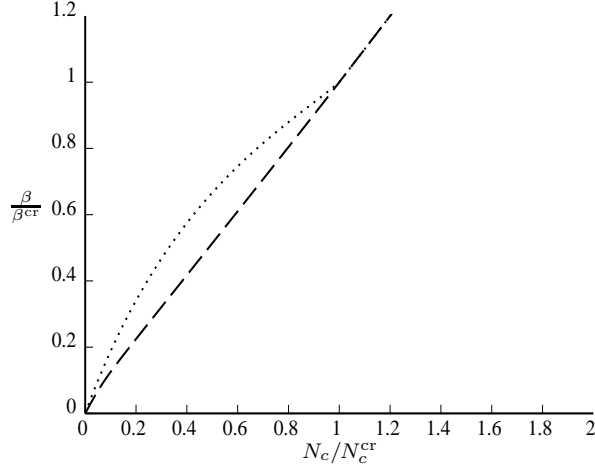


FIG. 5. *Approximation 1.* Reduced stiffness as a function of N_c/N_c^{cr} . The three curves below N_c^{cr} cannot be distinguished on this scale. The dotted curve corresponds to an extra low value of ρ_0 , just to show that below N_c^{cr} the curves deviate from a straight line.